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LETTER TO THE EDITOR

Directed recursive models for fractal growth

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Abstract. Fractals constructed by recursion processes are introduced to model growth phenomena. These fractals are simultaneously *directed and self-similar* in analogy with patterns growing under diffusion-limited conditions. The multifractal nature of the harmonic measure associated with Laplacian interfaces is qualitatively interpreted using the models. Calculation of the largest singularity exponent allows us to make conclusions about the behaviour of diffusion-limited aggregates.

The growth of objects with fractal geometry [1-4] is a common phenomenon in nature. In many cases the reason for the fractal properties of the interface is the instability of the process due to the presence of a non-local field satisfying the Laplace equation. This kind of growth typically leads to branching fractal structures with deep fjords between the branches. The most studied examples include electrodeposition [5, 6], crystallisation [7, 8], dielectric breakdown [9] and viscous fingering [10, 11]. Although a considerable amount of information has accumulated about these processes [4, 12-15], the basic phenomenon still lacks theoretical description based on first principles. One way of understanding fractal growth is by building deterministic models and in the present letter we undertake this approach.

When designing a model for fractal growth one has to take into account some of the already known features of the corresponding real structures. Studies of the diffusion-limited aggregation (DLA) model of Witten and Sander [16] and the related experimental systems [5-11] have greatly contributed to our knowledge of these properties. The structure of DLA clusters has been shown to exhibit two kinds of anisotropies. The overall shape of clusters generated on a square lattice is initially approximately circular, but for very large sizes it crosses over into a cross-like pattern [17, 18]. Investigations of the angle-dependent correlations revealed another type of anisotropy: the exponent describing the decay of the density correlations within the DLA clusters is smaller in the radial direction than in the tangential one [18]. As far as concerning the multifractal nature [19-23] of the growth probability distribution of Laplacian patterns numerical [24-26] and experimental [27, 28] evidence supports that the growth velocity of the interface can be described in terms of a fractal measure.

Deterministic or other models based on constructing a fractal recursively represent a useful tool in studying fractal growth because they allow for exact treatment of a number of properties [29, 30]. For example, the scaling of the cluster size distribution

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in diffusion-limited deposits observed in the simulations holds exactly in a deterministic model for DLA [29]. Recently branching Julia sets were proposed as deterministic structures with a harmonic measure on them analogous to the growth probability distribution of Laplacian patterns [31]. Numerical integration of the Laplace equation represents a further method of description of fractal growth by deterministic approaches [32, 33].

An important common feature of growth phenomena is that the motion of the interface starts from a small bounded region (or in some other cases from a hyperplane) and, correspondingly, the growth takes place in a direction pointing away from the seed configuration (or from the initial surface). This behaviour is obvious from simply looking at the computer-generated or experimental structures [4]. The branches tend to grow outward, and practically there are no branches developing in a direction of the seed. Thus, there exists a *directedness* inherent in Laplacian patterns due to this simple geometrical constraint. The above-mentioned angular dependence of the density correlations is one of the ways this property is manifested.

The directedness of the patterns raises the question of self-similarity of such structures. In other words: can directed clusters be self-similar? When the standard recursive methods of generating a deterministic fractal are applied [1] this problem does not arise. In the case of using a generator which leads to branching structures we can observe that the branches are turned with a given angle at each step of the iteration procedure resulting in branches directed in all possible directions, moreover, in the appearence of unphysical spirals.

The above undesirable effect is avoided in the model introduced in this letter. In the construction to be described below the units of the previous stage are replaced with an appropriately rotated and reflected (mirror image) version of the generating configuration. The model is demonstrated in figure 1. The first stage ($k=1$) is the generator: a simple branching structure made of three units (intervals of the same length). At the next stages each of the units obtained at the previous steps are replaced by the $k=1$ configuration while simultaneously obeying the following rules: (a) none of the branches should point in a direction below the horizontal; (b) no branches are allowed to overlap or touch each other. These rules can be satisfied uniquely by replacing a unit with either the generating configuration or with its mirror image. Figures 1(b) and 1(c) show the third and the seventh stages of the construction, respectively. The mathematical fractal is obtained in the $k\to\infty$ limit. For comparison a large branch of a DLA cluster is also displayed (figure 1(d)).

The fractal dimension of the above construction is

$$D=\ln 3/\ln 2\simeq 1.585. \tag{1}$$

This value is smaller than $D\simeq 1.7$ generally associated with diffusion-limited patterns. However, our goal is not to construct a model with $D\simeq 1.7$ (this can also be done), but to create a deterministic structure which possesses the main qualitative features of DLA clusters. In the present approach the angle θ between the vertical branch and the branch growing out to the right in the genrating configuration is an adjustable parameter. Varying the angle θ generates analogous, but different looking configurations (figure 2) having the same fractal dimension $D=\ln 3/\ln 2$.

The model illustrated in figure 1 can be generalised to include *randomness*. If the branching angle θ is smaller than or equal to $\pi/4$, two consecutive left or right turns (off the vertical direction) can be made without violating the condition of growing upward. This version of the model is degenerate in the sense that usually there is more

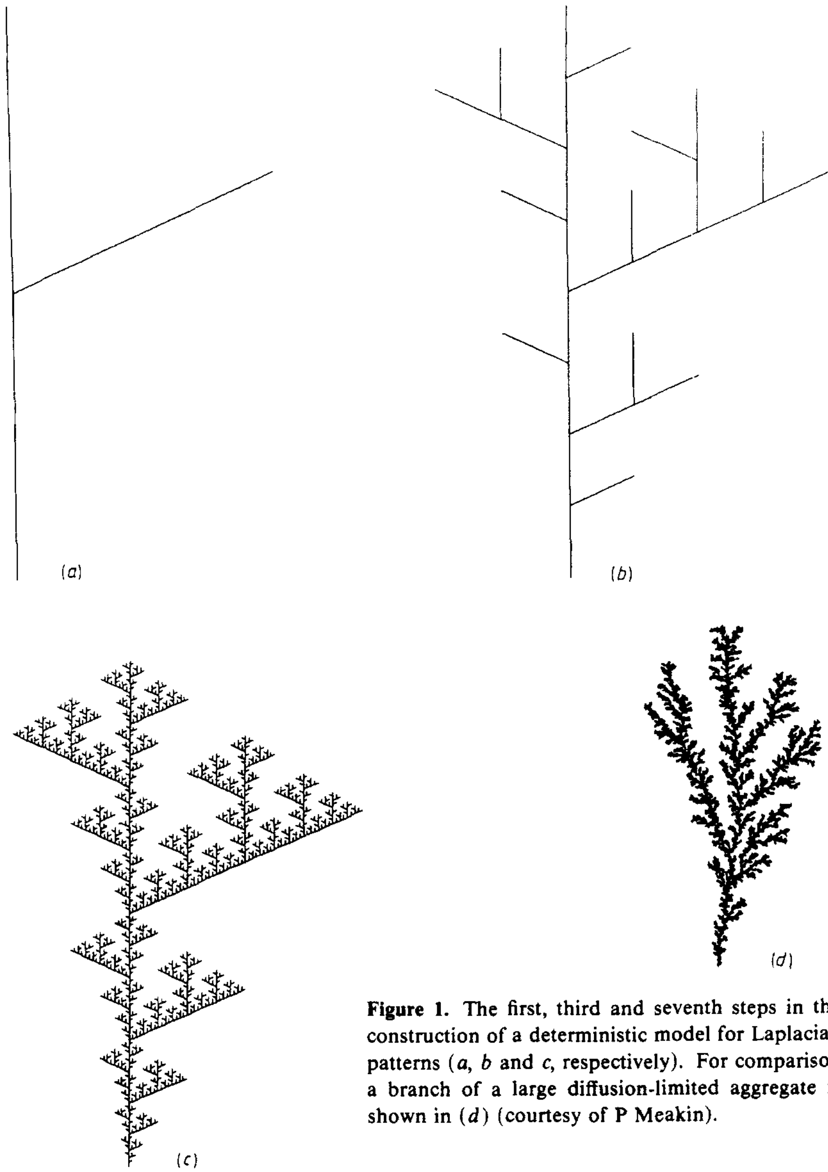

Figure 1. The first, third and seventh steps in the construction of a deterministic model for Laplacian patterns (a, b and c, respectively). For comparison a branch of a large diffusion-limited aggregate is shown in (d) (courtesy of P Meakin).

than one choice for the direction of the branches satisfying the rules (a) and (b). An example with $\theta=\pi/4$ generated by choosing randomly from the allowed directions when replacing a unit is shown in figure 3. These patterns seem to resemble the DLA branch displayed in figure 1(d) even more than the deterministic construction.

It is important to note that the fractals introduced in this letter have the following special property. Since they are constructed using a recursion procedure they are bound to be self-similar. On the other hand, they are directed: there is a well defined direction which can be attributed to the structure. Our model, however, is rather different from the commonly known examples for directed clusters, which are typically self-affine. For instance, the directed percolation clusters and the lattice animals are characterised by two scaling exponents, i.e., depending on the direction in which the

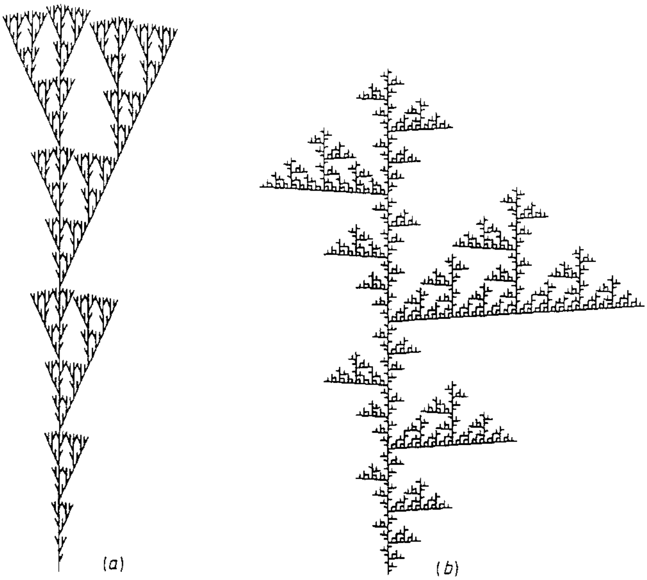

Figure 2. The same type of fractal as shown in figure 1 for two different branching angles. Both structures have the same fractal dimension $D = \ln 3/\ln 2 \simeq 1.585$.

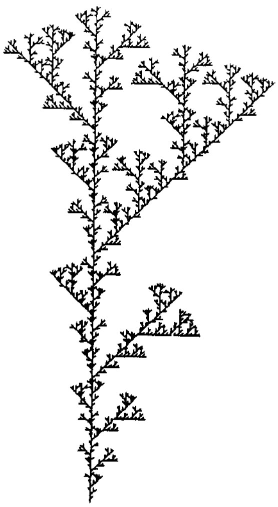

Figure 3. Random recursive model for branching angle less than $\pi/4$.

extension of the cluster is measured, their linear size grows according to different powers of the number of particles in them (see e.g. [34]). This is the property which indicates the self-affinity of such clusters, in contrast to the models introduced here.

Let us next analyse the model displayed in figure 1 from the point of view of its multifractal behaviour. Imagine that the structure is made of a perfect conductor and is electrically charged. Then the electric field (or the density of charge carriers) at the surface is called the harmonic measure corresponding to the given configuration. It is the gradient of a scalar field satisfying the Laplace equation with appropriate boundary conditions, and as such is in complete analogy with the growth probability or interface velocity distribution of Laplacian patterns.

A close inspection of, e.g., figure 2(*b*) shows that the structure is made of two kinds of almost closed loops being open to a different degree. (In the following we shall simply use the word loop for these imperfectly closed parts of the structure and the word opening for the entrance of these loops.) This observation makes it easier to understand what is the mechanism which produces the multifractal nature of the harmonic measure of DLA clusters. Each time we get deeper into the pattern and enter a loop, the field is decreased (screened) by a given factor λ_1 or λ_2 (depending which type of loop we enter). As a result, the strength of the field is determined by a *multiplicative process* with different weights λ_1 and λ_2 which is well known to lead to fractal measures [19, 23].

The above qualitative picture can be made more quantitative by noticing that the openings of the consecutive loops of the same type produce two characteristic cones with the corresponding angles φ_1 and φ_2 as demonstrated in figure 4(*a*). Next we assume that the ragged interface between the fractal and the cone-shaped empty regions can be approximated by a smooth surface such as in the probability scaling theory of DLA [35, 36]. However, in the present case instead of the convex shape with a tip we treat an incision having the shape of a cone. Nevertheless, we shall use the term tip for the place where the smoothed-out interface has a sharp turn.

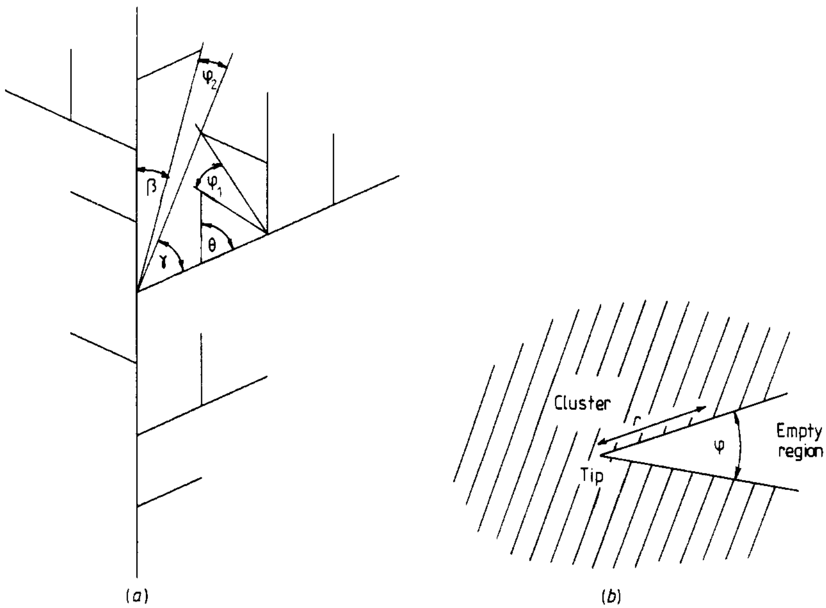

Figure 4. The angles depicted in this figure are used in the text to calculate the largest singularity exponent α_{max} of the harmonic measure associated with the fractal in figure 1.

The potential along the above-described cones is constant and we expect that its gradient (the electric field playing the role of the measure defined on the fractal) goes to zero as some power of the distance from the tip. The situation is analous to the picture used in the probability scaling approach [35, 36]; however, in our case φ_1 and

φ_2 are smaller than $\pi/2$. Thus, we can make use of the known solution of the simplified problem shown in figure 4(b):

$$\nabla\phi(r,\varphi)=\frac{C\pi}{2\varphi}r^{\pi/\varphi-1} \tag{2}$$

where ϕ denotes the electric potential, C is a constant and r is the distance from the tip. Let us now assume that the structure is covered by boxes of linear size ε. Then the amount of measure in a box consisting of a tip of angle φ is given by

$$m(\varepsilon)\sim\varepsilon^{\pi/\varphi}. \tag{3}$$

This expression is equivalent to a singularity exponent

$$\alpha=\frac{\ln m(\varepsilon)}{\ln\varepsilon}=\pi/\varphi. \tag{4}$$

As we saw, there are two characteristic angles and the smaller one gives the largest α. In the limit $\varphi\to 0$ we have $\alpha_{max}\to\infty$. One can find α_{max}(min), the smallest value (as a function of θ) of α_{max} from the condition $\varphi_1=\varphi_2$. This relation holds for a well defined θ which can be determined from the equations (figure 4(a))

$$\begin{aligned}\frac{\sin\gamma}{\sin[(\pi/4-(\gamma-\beta)/2]}&=2\\ \frac{\sin\beta}{\sin[(\pi/4)+(\gamma-\beta)/2]}&=\frac{1}{3}\end{aligned} \tag{5}$$

using $\varphi_1=\pi/2-\theta$, $\varphi_2=\theta-(\gamma-\beta)$ and the condition $\varphi_1=\varphi_2$ resulting in $\theta=\pi/4+(\gamma+\beta)/2$. The above equation can easily be solved numerically, and gives $\alpha\simeq 1.005$ and $\beta\simeq 0.305$. Consequently, the smallest α_{max} corresponds to $\varphi\simeq 0.131$ giving $\alpha_{max}(\min)\simeq 24$. This value is of the same order as α_{max}(DLA) $\simeq 9$, but is about 2.8 times larger.

It is an interesting fact that according to (4) $\alpha_{max}\simeq 9$ corresponds to $\varphi\simeq 0.35$. This value is quite close to $\varphi'\simeq 0.3$, where the tangential correlation function (describing the correlations in a layer at a given distance from the origin) of large off-lattice DLA clusters has its minimum. This minimum can be associated with the average angle of the empty region between the large branches of diffusion-limited aggregates, or with the cone angle in our approach. The coincidence supports the picture which was used in this letter, namely that α_{max} is determined by the characteristic angle of the deep cone-shaped fjords inside the aggregates. In addition, the relatively low value $\alpha_{max}\simeq 9$ indicates that diffusion-limited growth tends to take place in a way which minimises α_{max}.

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